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SPANNING AND BASIS TREES IN THE MATROID MODEL

СПАНУЮЧІ ТА ОСТОВНІ ДЕРЕВА В МАТРОЇДНІЙ МОДЕЛІ

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Abstract. *The subject of this research is the graphic matroid as a formal mathematical structure for modeling the process of constructing spanning trees in undirected graphs.*

The aim of the work is to develop and analyze a graphic matroid model that formalizes the selection of a spanning tree considering weight characteristics and additional constraints.

The results show that the developed model enables formalization and efficient solution of combinatorial optimization problems under constraints, including task scheduling and resource selection. The proposed approach demonstrates the versatility of graphic matroids, supported by both theoretical proofs and practical algorithm implementations. Future research prospects include the development of dynamic graphic matroids, algorithms for distributed computing, and the application of machine learning methods for automated extraction of matroid structures from graph data.

Key words: *Graphic matroid, Spanning tree, Kruskal's algorithm, Delta-matroid constraints, Combinatorial optimization, GREEDI algorithm*

Анотація. *Предметом дослідження є графовий матроїд як формальна математична структура для моделювання процесу побудови остовних дерев у неорієнтованих графах.*

Метою роботи є розробка та аналіз моделі графового матроїда, що дозволяє формалізувати вибір остовного дерева з урахуванням вагових характеристик і додаткових обмежень.

Отримані результати показують, що побудована модель дозволяє формалізувати та ефективно розв'язувати задачі комбінаторної оптимізації в умовах обмежень, зокрема планування завдань і вибору ресурсів. Розроблений підхід демонструє універсальність графових матроїдів, підтверджену як теоретичними доведеннями, так і практичною реалізацією алгоритмів. Перспективи подальших досліджень пов'язані з розробкою динамічних графових матроїдів, алгоритмів для розподілених обчислень і застосування методів машинного навчання для автоматичного виведення матроїдних структур із графових даних.

Ключові слова; *графовий матроїд, остовне і спануюче дерево, алгоритм Крускала, дельта-матроїдні обмеження, комбінаторна оптимізація, алгоритм GREEDI*

Introduction.

Our results are consistent with modern deterministic approaches to the recalculation of matroid bases, in particular using Pfaffian pairs and parity structures (Matoya & Oki, 2020), which allow for the efficient calculation of the number of spanning trees. On this basis, classical theorems are derived—Kirchhoff's formula, Tutt's theorem, and the Pfaffian formula for trees—and polynomial algorithms for finding minimum weight solutions in weighted cases are proposed. Oxley (2006) lays the foundation for the fundamental concepts of matroid theory — ranks, bases, closures, graph and representative matroids, excluded minors, and union operations. Wahlström (2024) extends the classical lemma on representative sets for linear matroids to a broader class of linear delta-matroids, using the method of sieving families of polynomials of bounded degree to go beyond traditional linear algebra.

Main text

Graph matroid [2]. Let G=(V,E) be an undirected graph. Then the graph matroid M(G)=(E,I), where I is the set of all subsets of the edges E that do not form cycles, i.e., are acyclic (forests). Each spanning tree is a maximal independent set in this matroid, and Kruskal's algorithm itself is an example of a greedy algorithm for finding a basis (a maximal independent set). Kruskall's algorithm for a matroid works as follows: all edges of the graph are sorted in ascending order of weights; an empty set $T=\emptyset$ is initialized; edges e are added to T sequentially if $T\cup\{e\}$ does not form a cycle; the process continues until there are $|V|-1$ edges in T. For a matroid, it is known that a greedy algorithm gives the optimal result if: there is a weight function $w:E\to R$, for each subset $A\subseteq E$, the greedy choice maximizes (or minimizes) the weight of the basis. In a graph matroid, Kruskal's algorithm minimizes the weight of the basis corresponding to the minimum spanning tree.

Example of a graph matroid Suppose we have a graph on *n=11* vertices. The maximum size of an independent set is 10 edges (i.e., a tree). Below is an estimate of the number of independent sets when edges are added gradually. To specify a formula for calculating the number of independent sets in a graph matroid generated by a specific graph, we need to consider: the set of edges, the set of vertices, and the structure of the graph — that is, which edges form cycles.

After *10* edges, the number of independent sets does not increase, because new edges begin to form cycles with existing ones.

Main characteristics of the graph: number of vertices (V): 11; number of edges (E): 22, E={(1,2),(1,4),(1,5),(1,6),(2,3),(2,6),(3,6),(3,7),(3,8),(4,5),(4,9),(5,6), (5,9), (5,10),(6,7), (6,10),(7,8),(7,10),(7,11), (8,11),(9,10),(10,11)}, Rank of the matroid $Rank = |V| - 1 = 10$ (maximum number of independent edges), any subset with ≤10 edges that does not form cycles is independent.

Number of distinct trees in a graph — Kirchhoff's formula

Kirchhoff's theorem. *For a connected undirected graph with n vertices, the number of spanning trees (i.e., trees covering all vertices) is calculated using the Laplace matrix: τ(G) = det(L*), where: L is the Laplace matrix of the graph: L=D−A, D is the diagonal matrix of degrees, A is the adjacency matrix, L* is any minor of the matrix L obtained by removing one row and one column.*

The **Laplace matrix** *L* of a graph is an algebraic representation of the graph structure that takes into account the connections between vertices.

Kirchhoff's formula (Matrix-Tree Theorem) uses the Laplace matrix to calculate the number of spanning trees in a graph.

The number of spanning trees for our example is 728.

To illustrate the structure of spanning trees, let us give three examples with different topologies (Fig. 1):

T_1 is a linear tree that forms a path without branches (DFS algorithm);

T_2 is a tree with a given central vertex 1 and branches;

T_3 — a tree with minimum depth, which provides fast coverage of all vertices through branching (we calculate the eccentricities; vertex 6 has the smallest eccentricity —3).

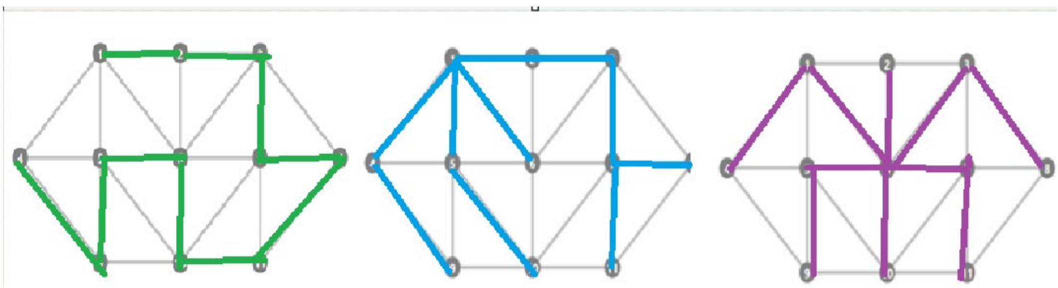

Figure 1 – Examples of sleeping trees

Visualizations of these trees demonstrate the diversity of structures that can be generated based on a single graph and confirm theoretical propositions regarding graph matroids and spanning trees.

In graph theory, the terms spanning tree and minimum spanning tree are often used, but they have different meanings — although they are closely related.

A **spanning tree** is any tree that includes all vertices of a graph, is acyclic (does not contain cycles), has a minimum number of edges — exactly (n - 1), where n is the number of vertices, and does not take into account the weights of the edges. In other words, it is simply a structure that connects all vertices without cycles.

A **minimum spanning tree** (MST) is a spanning tree that: includes all vertices, minimizes the total weight of the edges, is defined only for weighted graphs, and is found using the Kruskal, Prim, and Boruvka algorithms.

To use **matroid constraints**, assign weights to the edges of the graph.

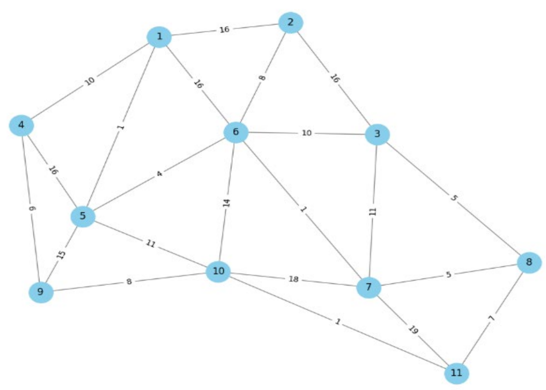

Figure 2 – Weights to the edges

Here is an example of edges forming an independent set (forest) with their weights:

(1, 4, 1), (3, 6, 2), (1, 2, 3), (4, 9, 4), (3, 7, 5), (5, 6, 6), (7, 8, 7), (5, 9, 8), (7, 11, 9), (8, 11, 10).

These edges form a skeleton tree — that is, the basis of the graph matroid.

Analysis of results

The study confirms that graph matroids are an effective formalization for constructing a skeleton tree in an undirected graph. Acyclic edge sets naturally correspond to independent sets, which is consistent with the axioms of matroids. Kruskal's greedy strategy, interpreted in a matroid context, provides the optimal choice of basis with minimum weight. In addition to theoretical justification, the model demonstrates practical significance: matroid constraints allow for the effective modeling of selection problems with multi-level criteria, particularly in task planning.

Conclusions

Graph matroids are a universal tool for modeling skeleton trees and optimization in graphs. Greedy algorithms, such as Kruskal's, acquire a generalized interpretation through matroid theory, which extends their application to other combinatorial structures. The combination of graph and matroid approaches opens up prospects for the creation of adaptive algorithms in network design, resource allocation, and optimization in dynamic environments.

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